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Department of Mathematics and Statistics
American University of Sharjah
Final Exam – Fall 2019
MTH 205-Differential Equations

Date: Sunday, December 15, 2019 **Time: 2pm to 4pm**

Student Name	Student ID Number
Aya Tarek	78806

Instructor Name	Class Time
Ayman Badawi	M, W : 11-12:15

1. ***Do not open*** *this exam until you are told to begin.*
2. ***No questions are allowed during the examination.***
3. *This exam has 8 pages + this cover exam page + Laplace Formula Sheet.*
4. *Do not separate the pages of the exam.*
5. *Scientific calculators are allowed.*
6. ***Turn off all cell phones and remove all headphones.***
7. *Take off your cap.*
8. ***No communication of any kind is allowed during the examination***
9. ***If you are found wearing a smart watch or holding a mobile phone at any point during the exam then it will be considered an academic violation and will be reported to the dean's office.***

Student signature: ______________

Final Exam , MTH 205 , Fall 2019

Ayman Badawi

QUESTION 1. (i) **(3 points)** Find the values of the constants $a, \cancel{b}^{k}, c$ which makes the differential equation $(12x^2y - aye^{cx})dx + (kx^3 - e^{3x})dy$ exact (**DO NOT SOLVE IT**)

$F_{xy} = F_{yx}$

$F_{xy} = 12x^2 - ae^{cx}$

$F_{yx} = 3kx^2 - 3e^{3x}$

$12x^2 - ae^{cx} = 3kx^2 - 3e^{3x}$

$12 = 3k \qquad \cancel{-}ae^{cx} = \cancel{-}3e^{3x}$

$\boxed{k = 4} \qquad \boxed{a = 3}$

$\boxed{c = 3}$

(ii) **(6 points)** Stare really good at the following diff. equation $\frac{dy}{dx} = \frac{y^3}{x^3 - xy^2}$, change it to Bernoulli and solve it.

$\frac{dx}{dy} = \frac{x^3 - xy^2}{y^3}$

$x' = \frac{1}{y^3}x^3 - \frac{1}{y}x$

$x' + \frac{1}{y}x = \frac{1}{y^3}x^3$

$v = x^{1-3} = x^{-2}$

$v' + (-2) \times \frac{1}{y}v = (-2)\frac{1}{y^3}$

$v' - \frac{2}{y}v = -\frac{2}{y^3}$

$I = e^{\int -\frac{2}{y}dy} = e^{-2\ln y}$
$= \frac{1}{y^2}$

$v = \frac{\int \frac{1}{y^2} \times -\frac{2}{y^3}\,dy}{\frac{1}{y^2}}$

$v = \frac{\int \frac{-2}{y^5}\,dy}{\frac{1}{y^2}}$

$v = \frac{\frac{1}{2}y^{-4} + c}{\frac{1}{y^2}}$

$v = \frac{1}{2}y^{-2} + y^2c$

$x = \left(\frac{1}{2}y^{-2} + y^2c\right)^{-\frac{1}{2}}$

QUESTION 2. (8 points) Use Laplace to solve the differential equation :

$$y'(t) = e^{3t} + \int_0^t 4y(u)\,du,\ y(0) = 0$$

$\int 4y(u)\,du$

$4 * y(t)$

$y'(t) = e^{3t} + 4 * y(t)$

$\mathcal{L}(y'(t)) = \mathcal{L}(e^{3t}) + \mathcal{L}(4 * y(t))$

$sY(s) - y(0) = \frac{1}{s-3} + \frac{4Y(s)}{s}$

$sY(s) - \frac{4}{s}Y(s) = \frac{1}{s-3}$

$Y(s)\left[s - \frac{4}{s}\right] = \frac{1}{s-3}$

$Y(s) = \frac{1}{(s-3)} \times \frac{s}{(s^2-4)}$

$Y(s) = \frac{s}{(s-3)(s-2)(s+2)}$

$\frac{s}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+2}$

$s = 3 \quad s = 2 \quad s = -2$

$A = \frac{3}{5} \quad B = \frac{-1}{2} \quad C = -\frac{1}{10}$

$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{3/5}{s-3} + \frac{1/2}{s-2} - \frac{1/10}{s+2}\right\}$

$y(t) = \frac{3}{5}e^{3t} - \frac{1}{2}e^{2t} - \frac{1}{10}e^{-2t}$

QUESTION 3. (10 points) Imagine a company is making fake-sweet-drink(only water and sugar). The Tank has a capacity of 700 Liters. Initially, it contains 250 Liters of brine (water and sugar) that contains 25 kg of sugar (i.e., assume $A(0) = 25$). A solution containing ~~4kg/L~~ 4L/min of sugar is pumped into the tank and solution is pumped out at 3~~kg/L~~ L/min.

(i) Find $A(t)$, the amount of sugar in the tank at time t.

$$\frac{dA}{dt} = \text{In} - \text{out}$$

$$A' = (4)(4) - C(t)3$$

$$C(t) = \frac{A}{250+(4-3)t}$$

$$A' = 16 - \frac{3A}{250+t}$$

$$A' + \frac{3}{250+t}A = 16$$

$$I = e^{\int \frac{3}{250+t}dt}$$

$$I = e^{+3\ln|250+t|}$$

$$I = (250+t)^3$$

$$A = \frac{\int (250+t)^3 \times 16}{(250+t)^3}$$

$$A = \frac{4(250+t)^4 + C}{(250+t)^3}$$

$$A(0) = \frac{4(250)^4 + C}{(250)^3} = 25$$

$$C = -1.52\times10^{10}$$

$$A(t) = \frac{4(250+t)^4 - 1.52\times10^{10}}{(250+t)^3}$$

0/0

(ii) Find the amount of sugar in the tank after 10 min.

$$A(10) = \frac{4(250+10)^4 - 1.52\times10^{10}}{(250)^3} = 195 \text{ kg}$$

$(250+10)^3$

(iii) When an overflow will occur?

$$250 + (4-3)t = 700$$

$$t = 450 \text{ mins}$$

QUESTION 4. (4 points) Consider the diff. equation $y' - 2xy = 0$, $y(0) = 1$. Now use power series to solve it (as explained in class), i.e., do the following:

(i) Find the recurrence formula. Calculate the coefficients of the first 5 terms (i.e., a_0, a_1, a_2, a_3, a_4)

$y = \sum_{n=0} a_n t^n \quad t = x$

$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots a_{n-1} t^{n-1} + a_n t^n + a_n t^{n+1}$

$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \ldots n a_n t^{n-1} + (n+1) a_{n+1} t^n$

$(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \ldots a a_n t^{n-1} + (n+1) a_{n+1} t^{n+1}) - 2t[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots a_n t^n + a_{n+1} t^{n+1}]$

$a_1 + (2a_2 - 2a_0)t + (3a_3 - 2a_1)t^2 + (4a_4 - 2a_2)t^3 \ldots (a_{n+1}(n+1) - 2a_{n-1})t^n = 0$

$a_1 = 0 \quad a_0 = 1 \quad 2a_2 - 2a_0 = 0 \quad a_2 = 1$

$n = 3 \rightarrow a_4 = \frac{2a_2}{4} = \frac{2 \times 1}{4} = \frac{1}{2}$

$(n+1)(a_{n+1}) - 2a_{n-1} = 0$

$a_{n+1} = \frac{2a_{n-1}}{n+1} \quad n \geq 1$

$\rightarrow y = 1 + t^2 + \frac{1}{2} t^4 \ldots$

$n = 1, \; a_2 = \frac{2a_0}{2} = \frac{2}{2} = 1$

$n = 2 \rightarrow a_3 = \frac{2a_1}{3} = \frac{0}{3} = 0$

$(a_0 = 1, \; a_1 = 0, \; a_2 = 1, \; a_3 = 0, \; a_4 = \frac{1}{2})$.

(ii) The power series in (1) converges to a well-known function, what is this function?(i.e., solve the diff. equation without using power series)

$I = e^{\int -2x} = e^{-x^2}$

$y = \frac{\int e^{-x^2} \times 0 \, dx}{e^{-x^2}} = \frac{0 + c}{e^{-x^2}}$

$y = c_1 e^{x^2}$

$y(0) = 1 \longrightarrow c = 1$

$y = e^{x^2}$

QUESTION 5. (7 points) Imagine that a 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is started in motion from the equilibrium position (i.e., $M(0) = 0$, note $M(t)$ is the motion of the spring, where small m is the mass) with an initial velocity of 1 m/sec in the upward direction (i.e., $M'(0) = -1$). Find the motion, $M(t)$, if the force due to air resistance is -90N.($g(gravity) = 9.8m/sec^2$)

mass = 10 kg

$L = 0.7$

$F = 10 \times 9.8 = 98 \, N$

$k = \frac{F}{L} = \frac{98}{0.7} = 140$

$F_{air} = -90$

-90

opposite direction

$M'' + \frac{-90}{10} M' + \frac{140}{10} M = 0$

$M'' - 9M' + 14M = 0$

$M = e^{mt}$

$m^2 - 9m + 14 = 0$

$m = 7 \quad m = 2$

$M = c_1 e^{7t} + c_2 e^{2t}$

$M(0) = c_1 + c_2 = 0$

$c_1 = -c_2$

$\rightarrow M = -\frac{1}{5} e^{7t} + \frac{1}{5} e^{2t}$

$M' = 7c_1 e^{7t} + 2c_2 e^{2t}$

$M'(0) = 7c_1 + 2c_2 = -1$

$7(-c_2) + 2c_2 = -1$

$c_2 = \frac{1}{5}$

$c_1 = -\frac{1}{5}$

QUESTION 6. (10 points) Use Laplace and solve the following system of Linear Diff. Equations:

$$x'(t) - y(t) = 0, x(0) = 2$$

$$y'(t) - x(t) = -t, y(0) = 1$$

$$sX(s) - x(0) - Y(s) = 0$$

$$sX(s) - Y(s) = 2 \quad \text{--- (1)}$$

$$sY(s) - y(0) - X(s) = -\frac{1}{s^2}$$

$$-X(s) + sY(s) = -\frac{1}{s^2} + 1 \rightarrow \frac{s^2-1}{s^2} \quad \text{--- (2)}$$

$$X(s) = \frac{\begin{vmatrix} 2 & -1 \\ \frac{s^2-1}{s^2} & s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{2s + \frac{s^2-1}{s^2}}{s^2-1}$$

$$X(s) = \frac{2s^3+s^2-1}{s^2(s^2-1)} = \frac{2s^{\cancel{3}}}{\cancel{s^2}(s^2-1)} + \frac{\cancel{s^2-1}}{s^2\cancel{(s^2-1)}}$$

$$X(s) = \frac{2s}{s^2-1} + \frac{1}{s^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2-1} + \frac{1}{s^2}\right\}$$

$$x(t) = 2\cosh(t) + t$$

$$Y(s) = \frac{\begin{vmatrix} s & 2 \\ -1 & \frac{s^2-1}{s^2} \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{\frac{s(s^2-1)}{s^2} + 2}{s^2-1} = \frac{s(s^2-1)+2s^2}{s^2(s^2-1)}$$

$$Y(s) = \frac{\cancel{s}\cancel{(s^2-1)}}{s^{\cancel{2}}\cancel{(s^2-1)}} + \frac{2\cancel{s^2}}{\cancel{s^2}(s^2-1)}$$

$$Y(s) = \frac{1}{s} + \frac{2}{s^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2-1}\right\} \Rightarrow y(t) = 1 + 2\sinh(t)$$

QUESTION 7. (6 points)

(i) $\ell\{\int_0^t e^{(t-u)}\cos(t-u)\sin(u)\,du\}$

$e^t \cos(t) * \sin(t)$

$$\mathcal{L}\{e^t\cos(t) * \sin(t)\}$$

$$= \frac{s-1}{(s-1)^2+1} \cdot \frac{1}{s^2+1}$$

(ii) Find $\ell^{-1}\{\frac{s(e^{-2s})}{(s+1)^2+4}\}$

$$u(t-2)\,\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+4}\right\} = \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

$$= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)$$

$$u(t-2)\left[e^{-(t-2)}\left(\cos(2t-4) - \frac{1}{2}\sin(2t-4)\right)\right]$$

QUESTION 8. (6 points) Solve for $y(t)$: $(cos(t) - t)y'' + (1 + sin(t))y' = 0$

$v = y'$

$v' = y''$

$(\cos(t) - t)v' + (1+\sin(t))v = 0$

$$v' + \frac{(1+\sin(t))}{(\cos(t)-t)}v = 0$$

$$I = e^{\int \frac{1+\sin(t)}{\cos(t)-t}}$$

$u = -(\cos(t) - t)$

$du = +\sin(t) + 1\,dt$

$$I = e^{\int \frac{1}{-u}du}$$

$$I = e^{-\ln|u|} = \frac{1}{-\cos(t)+t}$$

$$v = \frac{\int \frac{1}{t-\cos(t)} \times 0 \, dt}{\frac{1}{t-\cos(t)}}$$

$$v = \frac{0+c}{\frac{1}{t-\cos t}} \rightarrow c[t - \cos(t)]$$

$$y = \int ct - c\cos(t)\,dt$$

$$y = \frac{1}{2}ct^2 - c\sin t + c_1$$

$$y = ct^2 - c\sin t + c_1$$

QUESTION 9. (10 points)

(i) Solve for $y(t)$, $t^2y'' - 2ty' + 2y = 0$

$$y = t^m$$
$$y' = m\,t^{m-1}$$
$$y'' = (m^2 - m)\,t^{m-2}$$
$$t^m(m^2 - m - 2m + 2) = 0$$
$$m^2 - 3m + 2 = 0$$
$$m = 2 \text{ or } m = 1$$
$$y = C_1 t^2 + C_2 t$$

(ii) Use (1) and solve for $y(t)$: $t^2y'' - 2ty' + 2y = 2t^3e^t$

$$y = y_h + y_p$$
$$y_h = C_1 \underbrace{t^2}_{y_1} + C_2 \underbrace{t}_{y_2}$$
$$y_p = v_1 y_1 + v_2 y_2$$
$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = \frac{2t^3e^t}{t^2}$$

$$v_1' t^2 + v_2' t = 0$$
$$v_1' 2t + v_2' = 2te^t$$

$$W = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - 2t^2 = -t^2$$

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ 2te^t & 1 \end{vmatrix}}{-t^2} = \frac{-2\cancel{t^2}e^t}{-\cancel{t^2}}$$
$$v_1' = 2e^t$$
$$v_1 \int 2e^t\,dt = 2e^t$$

$$v_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & 2te^t \end{vmatrix}}{-t^2} = \frac{2t^{3}e^t}{-\cancel{t^2}}$$
$$v_2' = -2te^t$$
$$v_2 \int -2te^t\,dt$$

( )'	∫
$-2t$	e^t
-2	e^t
0	e^t

$$= -2te^t + 2e^t$$

$$y_p = (2e^t)(t^2) + (-2te^t + 2e^t)(t)$$
$$y_p = \cancel{2t^2e^t} - \cancel{2t^2e^t} + 2te^t$$
$$y_p = 2te^t$$

$$\Longrightarrow y = C_1t^2 + C_2t + 2te^t.$$

SHORT ANSWERS, JUST STARE well and Think

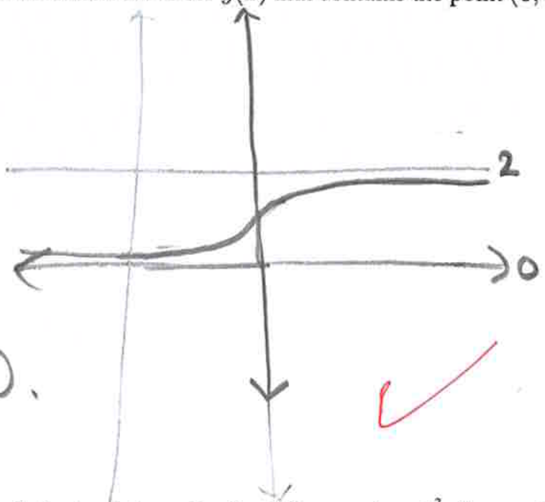
QUESTION 10. (i) **(3 points)** Draw the solution curve for $y(x)$ that contains the point $(0, 1.5)$ and find $lim_{x\to\infty}y(x)$, where $y' = -y^2 + 2y$.

$-y^2 + 2y = 0$

$y = 2$ or $y = 0$

$- - - - | + + + + + | - - - -$
$\quad 0 \qquad\quad 2$

$(0, 1.5)$ lies in $(0, 2)$.



as x goes ∞ → $y(x)$ will approach 2

$\lim_{x\to\infty} = 2$.

(ii) **(3 points)** Given y_1 and y_2 are two distinct solutions for the diff. equation $e^{x^2}y'' + cos(x)y = \frac{ln(x)}{1+x^3}$. Then one can quickly form a third solution $y_3 = \pi^2 y_1 + a y_2$ and a forth solution $y_4 = b y_1 + (e^2 + 1)y_2$. Find the values of the constants a, b.

$e^{x^2} y'' + \cos(x) y = \frac{\ln(x)}{1+x^3}$

$b = \pi^2$

$a = (e^2 + 1)$

0/3

(iii) **(4 points)** Solve the diff. equation $\frac{dy}{dx} = (\sqrt{y} + y)e^x(x^2 + 2x)$

$\int \frac{dy}{\sqrt{y} + y} = \int e^x(x^2 + 2x)\, dx$

$\int \frac{1}{(\sqrt{y})(1+\sqrt{y})} dy = x^2 e^x + C$

$u = 1 + \sqrt{y}$

$du = \frac{1}{2\sqrt{y}} dy$

$2 \int \frac{1}{u} du$

$2\ln|1+\sqrt{y}|$

$2\ln|1+\sqrt{y}| = x^2 e^x + C$

Faculty information

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